Problem 1

First we formulate the two programs into KAT.

```
while b do {
   p;
   while c do q
}
```

can be expressed as

$$(bp(cq)^*\bar{c})^*\bar{b} \tag{1}$$

and

```
if b then {
    p;
    while b∨c do {
        if c then q else p
    }
}
```

can be expressed as

$$bp((b+c)(cq+\bar{c}p))^*\overline{b+c}+\bar{b} \tag{2}$$

Because $\overline{b+c}=\bar{c}\bar{b}$ and

$$\begin{aligned}(b+c)(cq+\bar{c}p) &= bcq+b\bar{c}p+ccq+c\bar{c}p\\ &= bcq+b\bar{c}p+cq\\ &= (b+1)cq+b\bar{c}p\\ &= cq+\bar{c}bp,\end{aligned}$$

(2) can be transformed into

$$bp(cq+\bar{c}bp)^*\bar{c}\bar{b}+\bar{b} \tag{3}$$

Then we use the rule $1+pp^*=p^*$ on (1). We get

$$\begin{aligned}(bp(cq)^*\bar{c})^*\bar{b} &= (1+(bp(cq)^*\bar{c})(bp(cq)^*\bar{c})^*)\bar{b}\\ &= bp(cq)^*\bar{c}(bp(cq)^*\bar{c})^*\bar{b}+\bar{b} \qquad (4)\end{aligned}$$

This transformation correspond to loop unwinding operation, which turns the original program into

```
if b then {p; while c do q;
            while b do {
               p;
               while c do q
            }
}
```

Now we can see that $p(qp)^*=(pq)^*p$ may be applied on part in red in (4). We apply it and get

$$bp(cq)^*(\bar{c}bp(cq)^*)^*\bar{c}\bar{b}+\bar{b}$$

We see that we can apply this rule once more and get

$$bp\,((cq)^*\bar{c}\,bp)^*\,(cq)^*\bar{c}\,\bar{b} + \bar{b}$$

Now we can see the rule $(p^*q)^*p^* = (p + q)^*$ can be applied to the red part with substitutions $p = cq$ and $q = \bar{c}\,bp$. Apply it and we get

$$bp\,(cq + \bar{c}\,bp)^*\,\bar{c}\,\bar{b} + \bar{b}$$

which equals (3). This proves the programs are equal.